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Lattices, supersymmetry and Kähler fermions

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Abstract. It is shown that a graded extension of the space group of a (generalised) simple cubic lattice exists in any space dimension, D. The fermionic variables which arise admit a Kählerian interpretation. Each graded space group is a subgroup of a graded extension of the appropriate Euclidean group, $E(D)$. The relevance of this to the construction of lattice theories is discussed.

1. Introduction

There are good reasons for studying lattice approximations to supersymmetric theories. Two examples will suffice to make the point.

(a) It is too easy to write down a gauge invariant action and one would like further constraints to restrict the possibilities when model building. Supersymmetry would provide further constraints, but in order to produce a realistic particle spectrum it must be broken. Lattice regularisation provides the best framework known at present for the study of spontaneous breakdown of symmetry. It would be desirable, therefore, to be able to construct a lattice approximation to a supersymmetric continuum model.

(b) Difficulties are encountered when trying to describe fermions in lattice theories. If supersymmetry is taken seriously it should uniquely determine the form of the kinetic term for fermions, given the form of the bosonic term, as supersymmetry transformations relate the two.

There are difficulties in deciding what one means by a lattice approximation to a supersymmetric continuum theory. Several proposals have already been made, some of them using a Euclidean formulation (Dondi and Nicolai 1977, Nicolai 1978, Banks and Windey 1982), and others using a Hamiltonian formulation (Rittenberg and Yankielowicz 1982, Elitzur *et al* 1983). What all of these approaches have in common is that they are based on modifications of the graded Lie algebra of the super Lie group. This is not the only possible approach. Sakai and Sakamoto (1983) have a scheme based on the Nicolai (1980a, b) mapping, and yet another approach is described in this paper.

A lattice model which contains only bosonic fields is invariant under the action of some space group which depends on the type of lattice chosen. This space group is a subgroup of the Euclidean group $E(D)$ if the model is in D dimensions. It can be thought of as a discrete approximation to the Euclidean group. As the lattice spacing of the model is reduced, whilst at the same time the couplings are renomalised, this approximation becomes better and better until Euclidean invariance is restored (at least one hopes that this will happen). An immediate generalisation is possible: a

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lattice approximation to a supersymmetric model should be a lattice model which is invariant under a graded extension of the space group of the part of the action involving only the bosonic variables, this graded space group being a subgroup of a graded extension of the Euclidean group. Invariance under the complete graded Euclidean group should be recovered as the cut-off is removed. This is the point of view taken here. It is perhaps worth mentioning that some examples of discrete graded groups have been known for some time (Rogers 1981a, b).

After some algebraic preliminaries in § 2 it is shown in § 3 that discrete graded groups of the type described above do exist for 'cubic' lattices (cubic being suitably interpreted according to the dimension being considered). The preservation of the cubic symmetry ensures that the theory is invariant under Euclidean transformations in the naive continuum limit (Elitzur *et al* 1983). An interesting conclusion which is arrived at is that introducing a cubic lattice in superspace forces one to consider extended models whose fermions admit a Kählerian interpretation. This suggests that the language of differential forms may be useful in obtaining a lattice approximation to the continuum action—a prejudice which is reinforced by the observation that the de Rham, cubical and simplicial cohomology theories are isomorphic.

In § 4 it is shown how in the case of generalised nonlinear σ-models in two dimensions these ideas lead to a seemingly natural procedure for transcribing the continuum action to obtain a lattice action.

In § 5 a difficulty encountered when adopting this approach is explained.

Section 6 is the conclusion.

2. Algebraic preliminaries

Let C_L denote the complex Grassman (exterior) algebra over $\mathbb{C}^L$. Let M_L denote the set of sequences (Kostant 1977, Rogers 1980)

$$\{\mu \mid \mu=(\mu_1, \mu_2, \ldots, \mu_k),\ 1 \leqslant \mu_1 < \mu_2 < \ldots . < \mu_k \leqslant L,\ \mu_i \in \mathbb{Z} \text{ for } 1 \leqslant i \leqslant k\}.$$

Let Ω represent the empty sequence in M_L, and let (j) denote the sequence with just one element, j. Then there exists a basis of C_L of the form $\{\beta_\mu \mid \mu \in M_L\}$ with

$$\beta_\Omega = 1 \qquad \text{(the unit of } C_L)$$

$$\beta_\mu = \beta_{(\mu_1)}\beta_{(\mu_2)} \ldots \beta_{(\mu_k)} \qquad \text{for all } \mu \in M_L$$

$$\beta_{(i)}\beta_{(j)} = -\beta_{(j)}\beta_{(i)} \qquad \text{for } 1 \leqslant i, j \leqslant L.$$

(An infinite dimension Grassman algebra, C_∞, can be defined—see Rogers (1980) for details.)

If $b = \Sigma_{\mu \in M_L} \alpha_\mu \beta_\mu$ where $\alpha_\mu \in \mathbb{C}$ for all $\mu \in M_L$ then its conjugate, b^*, is defined to be $\Sigma_\mu \bar{\alpha}_\mu \beta_\mu^*$ where the bar denotes complex conjugation, and

$$\beta_\mu^* \equiv (\beta_{(\mu_1)}\beta_{(\mu_2)} \ldots \beta_{(\mu_k)})^* := \beta_{(\mu_k)}\beta_{(\mu_{k-1})} \ldots \beta_{(\mu_1)}.$$

An element $b \in C_L$ is called self-conjugate if and only if $b^* = b$. The set of self-conjugate elements, $\tilde{B}_L$, inherits the structure of a real vector space. It is not, however, a self-conjugate Grassman algebra for whilst

$$\beta_{(i)}^* = \beta_{(i)} \qquad \text{for all } i,\ 1 \leqslant i \leqslant L,$$

$$(\beta_{(i)}\beta_{(j)})^* = -\beta_{(i)}\beta_{(j)}.$$

By associating appropriate factors of the square root of minus one with the elements of the basis $\{\beta_\mu\}$ of C_L a self conjugate basis $\{\tilde{\beta}_\mu\}$ can be constructed which is also a basis for $\tilde{B}_L$

$$\tilde{B}_L = \left\{ \sum_{\mu \in M_L} \lambda_\mu \tilde{\beta}_\mu \middle| \lambda_\mu \in \mathbb{R} \right\}.$$

One has

$$C_L = C_{L,0} \oplus C_{L,1} \qquad \text{and} \qquad \tilde{B}_L = \tilde{B}_{L,0} \oplus \tilde{B}_{L,1}$$

where the subscript '0' indicates that elements of the space are linear combinations of basis elements with an even number of subscripts, or no subscripts (i.e. even elements), and the '1' indicates that the elements of the space are linear combinations of basis elements with an odd number of subscripts (i.e. odd elements).

It is convenient to introduce the objects

$$Z_L := \left\{ \sum_\mu n_\mu \tilde{\beta}_\mu \middle| n_\mu \in \mathbb{Z} \right\}, \qquad Z_L = Z_{L,0} \oplus Z_{L,1}.$$

3. Graded extensions of the space group of the simple cubic lattice

Consider, for the moment, two dimensions. In two dimensions a Dirac spinor has two complex components and the Clifford algebra is

$$\{\gamma_b, \gamma_c\} = -2\delta_{b,c}, \qquad 1 \leq b, c \leq 2.$$

A representation of this algebra is given by $\gamma_b = -\mathrm{i}\sigma_b$ (the σ's being the familiar Pauli matrices).

A possible graded extension of the subgroup of translations of E(2) is

$$(x_b, \theta_\alpha) \circ (y_b, \varepsilon_\alpha) := (x_b + y_b + \theta^\dagger \gamma_b \varepsilon - \varepsilon^\dagger \gamma_b \theta, \theta_\alpha + \varepsilon_\alpha)$$

where x_b, $y_b \in \tilde{B}_{L,0}$, $1 \leq b \leq 2$; and θ_α, $\varepsilon_\alpha \in C_{L,1}$, $1 \leq \alpha \leq 2$. A discrete subgroup is obtained by insisting that elements have the form

$$(am_b, (a/2)^{1/2}\{p_\alpha + \mathrm{i}q_\alpha\}), \qquad m_b \in Z_{L,0}, \qquad p_\alpha, q_\alpha \in Z_{L,1}.$$

Here 'a' is the lattice spacing. The multiplication law for the complete group is (dropping indices)

$$[\Lambda, (x, \theta)] \times [\Gamma, (y, \varepsilon)] := [\Lambda\Gamma, (x, \theta) \circ (R(\Lambda)y, \Lambda\varepsilon)].$$

Λ and Γ are elements of Pin(2). (Pin(2) bears the same relationship to O(2) as Spin (2) bears to SO(2); there is a projection mapping p: Pin(2) $\to$ O(2), see Curtis (1979) for details.) R is the vector representation of Pin(2). Now a rotation through 90° in the bosonic subspace corresponds to a rotation through only 45° in the fermionic subspace, consequently the set of elements of the form

$$[\Lambda', (am_b, (a/2)^{1/2}\{p_\alpha + \mathrm{i}q_\alpha\})]$$

where Λ' is an element of D_4' do *not* form a subgroup. ($D_4' = p^{-1}(D_4)$.)

This problem can be overcome by the introduction of a Pin(2) internal symmetry group. The Grassman variable θ_α acquires an extra label and can be thought of as a 2×2 matrix $(\Theta_{\alpha i})$. Now any 2×2 matrix can be written as a linear combination of the matrices $1(=I_2)$, $\Gamma_a(=\gamma_a)$, and $\Gamma_{12}(=\gamma_1\gamma_2=-i\sigma_3)$. The equation

$$\Theta_{\alpha i}=\theta_\mu(\Gamma_\mu)_{\alpha i} \tag{1}$$

defines the coefficients θ_μ (μ stands for a set of indices as usual). Under the action of the spinor rotation group $\Theta\mapsto\Lambda\Theta$, $\Lambda\in\mathrm{Pin}(2)$, and under the action of the internal symmetry group $\Theta\mapsto\Theta U^\dagger$, $U\in\mathrm{Pin}(2)$.

Under the action of the diagonal subgroup obtained by insisting that $\Lambda=U$, $\Theta\mapsto\Lambda\Theta\Lambda^\dagger$, and the coefficients θ_μ transform like tensors as their indices suggest. If one insists that $\Lambda=U\in\mathrm{D}_4'$ the external and internal symmetry transformations combine to produce 90° rotations (or multiples thereof) of the θ_μ's. Consequently the θ_μ's can be restricted to a discrete set of values and a discrete subgroup exists of the $N=4$ graded Euclidean group with Pin(2) internal symmetry group. N can be reduced to 2 by imposing a pseudo-Majorana condition:

$$\Theta=B\Theta^*B^{-1}$$

where B is a matrix with the property that

$$B\gamma_a^*B^{-1}=-\gamma_a.$$

In the Pauli representation $B=\gamma_1$ is a possible choice.

In D dimensions analogous results can be obtained by introducing a Pin(D) internal symmetry group to complement the Pin(D) (spinor) rotation group. In four dimensions this procedure leads to an $N=4$ theory.

The connection with Kähler fermions (Kähler 1962) can now be seen. The exterior algebra of differential forms in any dimension can be given the structure of a Clifford algebra (Kähler 1962, Atiyah 1970). Using this fact one can associate a spinor of the form given in equation (1) with a differential form, and the Dirac equation can be transcribed with the aid of the exterior derivative into an equation involving differential forms (Kähler 1962). (The situation is slightly more complicated in odd dimensions that in even dimensions because the irreducible representations of the Clifford algebra are not faithful.)

Elitzur *et al* (1983) have come to similar conclusions in 1+1 and 3+1 dimensions. Their approach rests on the Hamiltonian formalism and concentrates on the super Lie algebra rather than the group.

4. Two-dimensional nonlinear models

The pseudo-Majornana condition implies that a fermionic variable Ψ has the form

$$\begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix}, \qquad a, b\in C_{L,1}.$$

Introducing the column vector $\psi=\binom{a}{b}$, Ψ can be thought of as being composed of two column vectors ψ and $\sigma_1\psi^*$. Then

$$\mathrm{Tr}(\Psi^\dagger\Gamma_a\Theta)=\psi^\dagger\Gamma_a\theta-\theta^\dagger\Gamma_a\psi$$

where θ bears the same relationship to Θ as ψ bears to Ψ. In this new notation the subgroup of translations of the $N=2$ supersymmetry group in two dimensions has the composition law

$$(x_a, \theta_\alpha)\circ(y_a, \varepsilon_\alpha) = (x_a + y_a + \theta^\dagger\Gamma_a\varepsilon - \varepsilon^\dagger\Gamma_a\theta, \theta_\alpha + \varepsilon_\alpha).$$

Except for the fact that there are only two bosonic coordinates this is the composition law of the $N=1$ supersymmetry group in $3+1$ dimensions written in terms of Weyl spinors. The combinatoric properties of the representation theories are therefore the same, only the spin content differs, hence there is a two-dimensional analogue of the chiral multiplet:

$$A, \qquad \psi_\alpha := Q_\alpha A, \qquad F_{\alpha\beta} := Q_\alpha\psi_\beta = Q_\alpha Q_\beta A, \qquad \bar{Q}_\alpha A := 0.$$

Here the Q's are the generators of supersymmetry transformations. The action is (Wess and Zumino 1974)

$$\int d^2x[(\partial_a\bar{A})(\partial_a A) + \tfrac{1}{2}\psi^\dagger\overleftrightarrow{\not\partial}\psi + \tfrac{1}{2}\bar{F}_{\alpha\beta}F_{\alpha\beta}].$$

Now

$$\mathrm{Tr}(\Psi^\dagger\not\partial\Psi) = \psi^\dagger\overleftrightarrow{\not\partial}\psi = \psi^\dagger\not\partial\psi - (\partial_a\psi^\dagger)\gamma_a\psi.$$

So if one thinks of the fields as differential forms, and one introduces the exterior derivative d and the inner product of differential forms, the action can be rewritten as

$$\langle(d+d^\dagger)A, (d+d^\dagger)A\rangle + \langle\Psi, (d+d^\dagger)\Psi\rangle + \langle F, F\rangle, \qquad F := \tfrac{1}{2}\varepsilon_{\alpha\beta}F_{\alpha\beta},$$

which has the obvious lattice approximation (Becher 1981, Rabin 1982, Becher and Joos 1982)

$$\langle(\delta+\delta^\dagger)A, (\delta+\delta^\dagger)A\rangle + \langle\Psi, (\delta+\delta^\dagger)\Psi\rangle + \langle F, F\rangle$$

where δ is the coboundary operator defined for the square lattice and $\langle\,,\rangle$ is the inner product of cochains. Nonlinear models have a similar structure (Zumino 1979) and can be treated in an analogous fashion.

Supersymmetry is realised in these models through the following transformations:

$$A \mapsto A + i\sqrt{2}\varepsilon\psi + (\bar{\varepsilon}\tau_a\varepsilon)\partial_a A + \bar{\varepsilon}F\bar{\varepsilon} + i\sqrt{2}(\bar{\varepsilon}\tau_a\varepsilon)\partial_a(\bar{\varepsilon}\psi) + \tfrac{1}{2}(\bar{\varepsilon}\varepsilon)^2\partial^2 A,$$

$$\begin{aligned}\psi_\beta \mapsto \psi_\beta &- i\sqrt{2}(\tau_a\varepsilon)_\beta\partial_a A + i\sqrt{2}(\bar{\varepsilon}F)_\beta + (\bar{\varepsilon}\tau_a\varepsilon)\partial_a\psi_\beta + 2(\tau_a\varepsilon)_\beta\partial_a(\bar{\varepsilon}\psi)\\ &- i\sqrt{2}(\bar{\varepsilon}\tau_a\varepsilon)\partial_a(\bar{\varepsilon}F)_\beta - i\sqrt{2}\varepsilon_\beta(\bar{\varepsilon}\varepsilon)\partial^2 A - \tfrac{1}{2}(\bar{\varepsilon}\varepsilon)^2\partial^2\psi_\beta,\end{aligned}$$

$$\begin{aligned}F_{\beta\gamma} \mapsto F_{\beta\gamma} &- i\sqrt{2}(\tau_a\varepsilon)_\beta\partial_a\psi_\gamma + i\sqrt{2}(\tau_a\varepsilon)_\gamma\partial_a\psi_\beta - 2\varepsilon_\beta\varepsilon_\gamma\partial^2 A + (\tau_a\varepsilon)_\beta\partial_a(\bar{\varepsilon}F)_\gamma\\ &- (\tau_a\varepsilon)_\gamma\partial_a(\bar{\varepsilon}F)_\beta - i\sqrt{2}\varepsilon_\beta\varepsilon_\gamma\partial^2(\bar{\varepsilon}\psi) + \tfrac{1}{2}(\bar{\varepsilon}\varepsilon)^2\partial^2 F_{\beta\gamma}.\end{aligned}$$

Benn and Tucker (1983) have succeeded in writing various supersymmetric actions and their linearised transformation laws in terms of differential forms.

To complete the programme a realisation of the graded space group on the lattice fields must be given. Furthermore a prescription should be given for the construction of a third supermultiplet from two given supermultiplets. In order to be able to do this a definition of the product of two lattice Kähler fields is needed.

5. The Clifford product

A product is defined in cohomology theories which is called the cup product (in de Rham cohomology it is called the wedge product), but this product cannot be identified with the Clifford product which occurs in the albegra of the γ-matrices, and it is the Clifford product whose analogue is needed to be able to define the product of two lattice Kähler fields. In the continuum one has

$$\begin{aligned}\Theta\Psi &= (\theta_\phi 1 + \theta_a \Gamma_a + \theta_{12}\Gamma_{12})(\psi_\phi 1 + \psi_a\Gamma_a + \psi_{12}\Gamma_{12}) \\ &= (\theta_\phi\psi_\phi - \theta_a\psi_a - \theta_{12}\psi_{12})1 + (\theta_\phi\psi_a + \theta_a\psi_\phi - \theta_b\psi_{ba} - \theta_{ba}\psi_b)\Gamma_a \\ &\quad + (\theta_\phi\psi_{12} + \theta_{12}\psi_\phi + \varepsilon_{ab}\theta_a\psi_b)\Gamma_{12}.\end{aligned}$$

In a lattice analogue of this equation the coefficients of 1 would be associated with sites, the coefficients of Γ_1 and Γ_2 with links in the 1 and 2 directions respectively, and the coefficients of Γ_{12} with plaquettes. As this equation shows, one must be able, for instance, to compose two link variables associated with links pointing in the same direction to obtain a site variable. Becher and Joos (1982) have put forward a definition of such a product, but their definition suffers from the drawbacks that it is not associative and does not repsect the cubic symmetry of the lattice. These difficulties remain to be resolved.

6. Conclusion

By combining the idea that a lattice approximation to a continuum supersymmetric model should be invariant under the action of a discrete subgroup with some cohomology theory and the Kähler description of fermions some progress has been made towards the construction of a 'supersymmetric' lattice model. The chain of argument shows that insisting on a sort of cubic symmetry which guarantees Euclidean invariance in the naive continuum limit forces one to consider extended supersymmetries. The central problem in this way of viewing things is the definition of an analogue of the Clifford product which is appropriate to the lattice formulation.

Whilst there are still difficult problems to be resolved I believe that the approach outlined in this paper offers some hope of leading to a meaningful lattice approximation to a continuum model, and I think that it is worthy of further study.

After this work was completed a preprint was received at Edinburgh by Kostelecký and Rabin (1983). These authors have also hit upon the idea of using discrete subgroups of the supersymmetry group, but they consider the subgroup of translations only, and not its interaction with the subgroup of Lorentz transformations (they work in Minkowski space where the introduction of a lattice ruins invariance under boosts).

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